

# Capacity Planning in Stable Matching: An Application to School Choice

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In this work, we introduce the problem of jointly allocating a school capacity expansion (given a fixed budget) and finding the best allocation for the students in the expanded market. Given the computational complexity of the problem, we provide an integer quadratically-constrained programming formulation and study its linear reformulations. We also propose two heuristics: A greedy algorithm and an LP-based method. We empirically evaluate the performance of our approaches in a detailed computational study. We observe the practical superiority of the linearized model in comparison with its quadratic counterpart and we outline their computational limits. Finally, we use the Chilean school choice system data to empirically demonstrate the impact of capacity planning under stability conditions. Our results show that each additional school seat can benefit multiple students. In addition, depending on the decision-maker, our methodology can prioritize the assignment of previously unassigned students or improve the assignment of several students through improvement chains.

*Key words:* Stable Matching, Capacity Planning, Hospital/Resident problem, School Choice, Integer Programming

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## 1. Introduction

Centralized mechanisms are becoming the standard approach to solve several assignment problems. Examples include the allocation of students to schools, high-school graduates to colleges, residents to hospitals and refugees to cities. In most of these markets, a desirable property of the assignment is *stability*, which guarantees that no pair of agents has incentive to circumvent the match. As discussed in Roth (2002), finding a stable matching is crucial for the clearinghouse’s success, long-

term sustainability and also ensures some notion of fairness as it eliminates the so-called *justified-envy*.<sup>1</sup>

A common assumption in these markets is that capacities are fixed and known in advance. However, capacities are only a proxy of how many agents can be accommodated, and there is some flexibility to modify them in multiple settings. For instance, in refugee resettlement (Delacretaz et al. 2016, Andersson and Ehlers 2020), local authorities define how many refugees they are willing to receive, but they could increase their capacity given proper incentives. Similarly, in healthcare rationing (Pathak et al. 2020, Aziz and Brandl 2021), policy-makers can make additional investments to expand the amount of resources available. In school choice, administrators report how many open seats they have in each grade, based on their current enrollment and the size of their classrooms. However, they could switch classrooms of different sizes to modify the seats offered on each level. Finally, in college admissions, colleges may increase their capacities to admit all tied students competing for the last seat (Rios et al. 2021).

As the previous discussion illustrates, capacities may be flexible, and it may be natural to incorporate them as a decision to further improve the assignment process. Indeed, by jointly deciding capacities and the allocation, the clearinghouse can leverage its knowledge about agents' preferences to achieve different goals. On the one hand, one possible goal is to maximize *access*, i.e., to choose the capacity that maximizes the total number of agents being assigned. This objective is especially relevant in some settings, such as school choice, where the clearinghouse wants to ensure that it gives every student a seat in some school. On the other hand, the clearinghouse may wish to prioritize *improvement*, i.e., to enhance the assignment of high-priority students. This objective is common in merit-based settings such as college admissions and the hospital-resident problem. Therefore, in the setting with capacity expansion, there might be multiple *optimal assignments* due to the trade-off between *access* and *improvement*. This is particularly relevant, because in the vanilla version of the problem without capacity expansion, there is a unique optimal assignment and the aforementioned trade-off does not occur.

In this paper, we study how to jointly find a stable assignment and make capacity decisions. To accomplish this, we introduce a capacity planning model in the context of stable matching. We show that we can formulate the problem as a quadratically-constrained integer program, and we provide two linearizations to facilitate its solution. Given the complexity of the problem (Bobbio et al. 2022), we propose two natural heuristics: (i) a greedy approach, which increases the capacity of the school leading to the highest improvement; and (ii) a linear programming approach, which relaxes the stability constraints to make capacity decisions, and later considers the updated capacities to

<sup>1</sup> See Romm et al. (2020) for a discussion on the differences between *stability* and *no justified-envy*.

find a stable assignment using the Deferred Acceptance (DA) algorithm—the classic method for stable assignments. To assess the performance of our linearizations and heuristics, we perform an extensive computational study on a synthetic dataset. Finally, we use the Chilean school choice system data to show the potential benefits of embedding capacity decisions. Our results show that each additional seat can benefit multiple students. In addition, depending on how we define the objective, our methodology can prioritize the assignment of previously unassigned students or improve the assignment of several students through improvement chains. Our results show that the proposed heuristics are computationally efficient and achieve close-to-optimal solutions.

### 1.1. Contributions

Our work combines a variety of methodologies and makes several contributions that we now describe in detail.

*Problem formulation.* To capture the problem described above, we introduce a stylized model of a many-to-one matching market, in which the clearinghouse can make capacity planning decisions while simultaneously finding a student-optimal stable matching. We show theoretically and empirically that the clearinghouse can prioritize different goals when making capacity decisions. On one end, the clearinghouse may prioritize *access*, i.e., maximize the number of students assigned. On the other, the clearinghouse may prioritize *improvement*, i.e., maximize the preference of assignment of highly ranked students. Our model is flexible enough to accommodate any of these goals and, more importantly, to understand the trade-offs between them. Finally, we show that our model can be extended in several interesting directions, including the addition of costs to expand capacities, adding secured enrollment, planning classroom assignment to different grades, etc.

*Mathematical programming approach.* In the standard case with no capacity decisions, we first show that we can find the student-optimal stable matching by minimizing the sum of the students’ assignment preferences over the set of feasible stable matchings. In addition, when we include capacity decisions, we show that we can formulate the problem as an integer quadratically-constrained program (IQCP). We provide two different linearizations to improve the computational efficiency and we show that one has a better linear relaxation.

*Heuristics.* As shown in (Bobbio et al. 2022), the problem is NP-hard, motivating the design of heuristics that would provide good solutions in a reasonable computational time. In particular, we propose two new heuristics to solve the problem. Our first heuristic, called *Greedy*, assigns extra seats to the schools leading to the most significant change in the objective function. Our second heuristic, called *LPH*, proceeds in two steps by first solving the problem without stability constraints to find the allocation of extra seats and then finding the student-optimal stable matching conditional on the capacities defined in the first step. Our computational experiments show that both heuristics significantly reduce the time to find a close-to-optimal solution. Moreover, we find that LPH outperforms Greedy when the budget of extra seats increases.

*Empirical application.* To illustrate the benefits of embedding capacity decisions, we use data from the Chilean school choice system and we adapt our framework to solve the problem including all the specific features described in (Correa et al. 2022). First, we show that each additional seat can benefit multiple students. Second, in line with our theoretical results, we find that *access* and *improvement* can be prioritized depending on how unassigned students are penalized in the objective. Finally, our results show that the students’ matching is ameliorated even if we upper bound the total number of additional seats per school.

*Societal Impact.* Given these positive results, we are currently collaborating with the institution in charge of implementing the Chilean school choice system to test our framework in the field. Moreover, our model can be easily adapted to tackle other impactful questions. For instance, it can be used to optimally decide how to decrease capacities, as some school districts are experiencing large drops in their enrollments (Tucker 2022). Our model could also be used to optimally allocate tuition waivers under budget constraints, as in the case of Hungary’s college admissions system. These examples illustrate how our framework can answer policy-relevant questions in other settings.

The remainder of this paper is organized as follows. In Section 1.2, we provide a literature review. In Section 2, we introduce our model. In Section 3, we present an IQCP formulation, and we study two McCormick linearizations. In Section 4, we provide a detailed computational study on a synthetic dataset for the case when students have complete preference lists. In Section 5, we evaluate our framework using data from the Chilean school choice system. Finally, in Section 6, we conclude.

## 1.2. Related Work

Gale and Shapley (1962) introduce the well-known Deferred Acceptance (DA) algorithm, which finds a stable matching in polynomial time for any instance of the problem. Since then, the literature on stable matchings has extensively grown and has focused on multiple variants of the problem. For this reason, we focus on the most closely related literature, and we refer the interested reader to Manlove (2013) for a broader literature review.

*Mathematical programming formulations.* The first mathematical programming formulations of the stable matching problem were studied in Gusfield and Irving (1989), Vate (1989), Rothblum (1992) and Roth et al. (1993). Thereafter, Baiou and Balinski (2000) provide an exponential size linear programming formulation and prove that it coincides with the convex hull of the set of feasible stable matchings. Moreover, they give a polynomial-time separation algorithm. Kwanashie and Manlove (2014) provide an integer formulation of the problem when there are ties in the preference lists. Kojima et al. (2018) introduce a way to represent preferences and constraints to guarantee strategy-proofness. Ágoston et al. (2016) propose an integer model that incorporates

upper and lower quotas. Delorme et al. (2019) devise new mixed-integer programming formulations and pre-processing procedures. More recently, Ágoston et al. (2021) propose similar mathematical programs and use them to compare different policies to deal with ties. We contribute to this literature by adding capacity decisions in finding a stable matching.

*Capacity expansion.* To the best of our knowledge, our paper is the first to introduce the problem of optimal capacity planning in the context of stable matching. Closely related to our paper is (Bobbio et al. 2022), who studies the complexity of the problem and of other extensions. Specifically, the authors show that the decision version of the problem is NP-complete and inapproximable within a  $O(\sqrt{n})$  factor, where  $n$  is the number of students. Abe et al. (2022) study this problem and propose an alternative method to solve the capacity expansion problem. Specifically, their approach relies on an anytime method where the upper confidence tree (UCT) searches the space of capacity expansions. Finally, Nguyen and Vohra (2018) study the impact of capacity expansion when there are couples in the market. In particular, they show that the existence of a stable matching is guaranteed if the capacity of the market is expanded by at most 9 spots.

*School choice.* Starting with (Abdulkadiroğlu and Sönmez 2003), a large body of literature has studied different elements of the school choice problem, including the use of different mechanisms such as DA, Boston, and TTC (Abdulkadiroğlu et al. 2005, Pathak and Sönmez 2008, Abdulkadiroğlu et al. 2011); the use of different tie-breaking rules (Abdulkadiroğlu et al. 2009, Arnosti 2015, Ashlagi et al. 2019); the handling of multiple and potentially overlapping quotas (Kurata et al. 2017, Sönmez and Yenmez 2019); the addition of affirmative action policies (Ehlers 2010, Hafalir et al. 2013); and the implementation in many school districts and countries (Abdulkadiroğlu et al. 2005, Calsamiglia and Güell 2018, Correa et al. 2022). Within this literature, the closest papers to ours are those that combine the optimization of different objectives with finding a stable assignment. Caro et al. (2004) introduce an integer programming model to make school redistricting decisions. Shi (2016) proposes a convex optimization model to decide the assortment of schools to offer to each student to maximize the sum of utilities. Ashlagi and Shi (2016) present an optimization framework that allows them to find an assignment pursuing (the combination of) different objectives, such as average and min-max welfare. Finally, Bodoh-Creed (2020) presents an optimization model to find the best stable and incentive-compatible match that maximizes any combination of welfare, diversity, and prioritizing the allocation of students to their neighborhood school. Our paper contributes to this literature by introducing capacity planning in the problem of finding a stable matching, and showing its potential impact in the Chilean school choice system.

*Entry comparative static and strategy-proofness.* The design of a stable matching mechanism when the number of participants of one side is increased has already been investigated through the lens of game theory. In particular, for the stable marriage problem, this is known as the *entry*

*comparative static*. In (Kelso Jr and Crawford 1982, Gale and Sotomayor 1985, Roth and Sotomayor 1990), the authors prove, for example, that when a new woman is added to the instance, all men are matched to a *weakly better* partner. Recently, Kominers (2019) extended this result to the school admission problem. In (Balinski and Sönmez 1999), the authors show that the DA algorithm is invariant with respect to improvements of the students' position in the preference lists of the schools. The literature has also studied matching mechanisms that incentivize participants to reveal their true preferences; this is known as strategy-proofness. For instance, Sönmez (1997) proves that schools can manipulate the stable matching in their favor by falsely reporting a reduced capacity. Romm (2014) proves that manipulation is still possible even if the reported capacities are enforced during the admission process.

## 2. Model

We formalize the stable matching problem using school choice as an illustrating example.<sup>2</sup> Let  $\mathcal{S} = \{i_1, \dots, i_{|\mathcal{S}|}\}$  be the set of students, and let  $\mathcal{C} = \{j_1, \dots, j_{|\mathcal{C}|}\}$  be the set of schools.<sup>3</sup> Each student  $s \in \mathcal{S}$  reports a strict preference order  $\succ_s$  over the elements in  $\mathcal{C} \cup \{\emptyset\}$ . Note that we allow for  $\emptyset \succ_s c$  for some  $c \in \mathcal{C}$ , so students may not include all schools in their preference list. In a slight abuse of notation, we use  $|\succ_s|$  to represent the number of schools to which student  $s$  applies and prefers compared to being unassigned, and we use  $c' \succeq_s c$  to represent that either  $c' \succ c$  or that  $c' = c$ . On the other side of the market, each school  $c \in \mathcal{C}$  ranks each student according to a strict order  $\succ_c$ . Moreover, we assume that each school  $c \in \mathcal{C}$  offers a number of seats  $q_c \in \mathbb{Z}_+$  and we assume that  $\emptyset$  has infinite capacity.

Let  $\mathcal{E} \subseteq \mathcal{S} \times (\mathcal{C} \cup \{\emptyset\})$  be the set of feasible pairs, with  $(s, c) \in \mathcal{E}$  meaning that  $s$  applied and satisfies all requirements to be admissible at school  $c$ . We assume that  $(s, \emptyset) \in \mathcal{E}$  for all  $s \in \mathcal{S}$ . Then, a *matching* is an assignment  $\mu \subseteq \mathcal{E}$  such that each student is assigned to at most one school, and each school receives at most  $q_c$  students. In a slight abuse of notation, we use  $\mu(s) \in \mathcal{C} \cup \{\emptyset\}$  to represent the school of student  $s$  in the assignment  $\mu$ , with  $\mu(s) = \emptyset$  representing that  $s$  is unassigned in  $\mu$ . Similarly, we use  $\mu(c) \subseteq \mathcal{S}$  to represent the set of students assigned to  $c$  in  $\mu$ . A matching  $\mu$  is stable if it has no *blocking pairs*, i.e., there is no pair  $(s, c) \in \mathcal{E}$  that would prefer to be assigned to each other compared to their current assignment in  $\mu$ . Formally, we say that  $(s, c)$  is a *blocking pair* if the following two conditions are satisfied: (1) student  $s$  prefers school  $c$  over  $\mu(s) \in \mathcal{C} \cup \{\emptyset\}$ , and (2)  $|\mu(c)| < q_c$  or there exists  $s' \in \mu(c)$  such that  $s \succ_c s'$ , i.e.,  $c$  prefers  $s$  over  $s'$ .

<sup>2</sup> Notice that once we break ties, students are strictly ordered in each school (given their priority, random tie-breaker, and possibly the quotas the student is eligible). Hence, the school choice setting becomes similar to other two-sided stable matching problems such as college admissions, the hospital-resident problem, among others.

<sup>3</sup> To facilitate the exposition, we assume that all students belong to the same grade, e.g., pre-kindergarten.

Gale and Shapley (1962) show that, for any instance  $\Gamma = \langle \mathcal{S}, \mathcal{C}, \succ, \mathbf{q} \rangle$  of the problem, a stable matching can be found in polynomial time using the Deferred Acceptance algorithm. Moreover, by changing the proposing side, DA can be adapted to find (i) the *student-optimal* stable matching, i.e., the unique stable matching that is weakly preferred by all students, and (ii) the *school-optimal* stable matching, i.e., the unique stable-matching that all schools weakly prefer. In Appendix B.1, we formally describe the student-optimal DA algorithm.

Let  $r_{s,c}$  be the position of school  $c \in \mathcal{C}$  in  $s$ 's preference order, and let  $r_{s,\emptyset}$  be a parameter that represents the penalty to have student  $s$  unassigned.<sup>4</sup> In Lemma 1 we show that, for any instance of the problem, we can find the student-optimal stable assignment by solving an integer linear program whose objective is to minimize the sum of students' preference of assignment. The proof can be found in Appendix A.

LEMMA 1. *Given an instance  $\Gamma = \langle \mathcal{S}, \mathcal{C}, \succ, \mathbf{q} \rangle$ , finding the student-optimal stable matching is equivalent to solving the following integer program:*

$$\begin{aligned}
\min_x \quad & \sum_{(s,c) \in \mathcal{E}} r_{s,c} \cdot x_{s,c} \\
s.t. \quad & \sum_{c \in \mathcal{C} \cup \{\emptyset\}} x_{s,c} = 1, & \text{for all } s \in \mathcal{S}, \\
& \sum_{s \in \mathcal{S}} x_{s,c} \leq q_c, & \text{for all } c \in \mathcal{C}, \\
& q_c x_{s,c} + q_c \cdot \sum_{c' \succ_s c} x_{s,c'} + \sum_{s' \succ_c s} x_{s',c} \geq q_c, & \text{for all } (s,c) \in \mathcal{E}, \\
& x_{s,c} \in \{0, 1\}, & \text{for all } (s,c) \in \mathcal{E}.
\end{aligned} \tag{1}$$

Note that the Integer Program (1) can be solved in polynomial time using the reformulation and the separation algorithm proposed by Baiou and Balinski (2000).

## 2.1. Capacity Expansion

In this section, we adapt Formulation (1) to incorporate capacity expansion decisions. Let  $\mathbf{t} = \{t_c\}_{c \in \mathcal{C}} \in \mathbb{Z}_+^{\mathcal{C}}$  be the vector of additional seats allocated to each school  $c \in \mathcal{C}$ , and let  $\Gamma_{\mathbf{t}} = \langle \mathcal{S}, \mathcal{C}, \succ, \mathbf{q} + \mathbf{t} \rangle$  the instance of the problem in which the capacity of each school  $c$  is  $q_c + t_c$ .<sup>5</sup> The *capacity expansion problem* consists in finding how to allocate a budget  $B$ , and the corresponding student-optimal stable matching, leading to the minimum sum of preferences of assignment. This can be formalized as follows:

<sup>4</sup> Note that the penalty  $r_{s,\emptyset}$  may be different from the ranking of college  $\emptyset$  in student  $s$  preference list. As such, the penalty does not directly affect the stability condition.

<sup>5</sup> Notice that  $\Gamma_{\mathbf{0}}$  corresponds to the original instance  $\Gamma$  with no capacity expansion.

PROBLEM 1. Given a budget  $B \in \mathbb{Z}_+$  and an instance  $\Gamma$ , the *capacity expansion problem*, is the following:

$$\min_{\mathbf{t}, \mu} \left\{ \sum_{(s,c) \in \mu} r_{s,c} : \mathbf{t} \in \mathbb{Z}_+^{\mathcal{C}}, \sum_{c \in \mathcal{C}} t_c \leq B, \mu \text{ is a stable matching in instance } \Gamma_{\mathbf{t}} \right\}.$$

In other words, an optimal allocation of extra seats in Problem 1 leads to a student-optimal stable matching whose objective value is the best among all feasible capacity expansions. Note that Problem 1 may have multiple optimal allocations that lead to different student-optimal stable matchings with the same objective value (see Example 1 in Appendix A). Also, observe that an optimal allocation may not necessarily use the entire budget since the objective value may no longer improve (e.g., if we assign every student to their top preference). However, we can assume that the entire budget was used by arbitrarily assigning the remaining seats without affecting the objective value. Finally, note that we can easily adapt our model to capture different settings, including capacity reductions; allocations of tuition waivers; quotas; secured enrollment; arbitrary constraints on the extra seats per school, among others; see Appendix C for details.

## 2.2. Access vs. Improvement

It is important to notice that the solution to Problem 1 largely depends on the value of  $r_{s,\emptyset}$ , i.e., the *penalty* for having unassigned students. We emphasize  $r_{s,\emptyset}$  does not indicate the position of  $\emptyset$  in the ranking of  $s$  over schools, but a penalty value for being unassigned. One could expect that for larger penalty values, the optimal solution will prioritize *access* by matching initially unassigned students. In contrast, for smaller penalty values, the focus will be on *improvement* by prioritizing *chains of improvement* that enable multiple students to obtain a better assignment than the initial one (with no extra capacities). We remark that the initial matching is obtained by solving the Integer Program 1.

In the following results, we formalize the trade-off between access and improvement. Let  $\mu^B$  be an optimal assignment obtained with budget  $B \geq 0$ . In addition, given a budget  $B > 0$  and an optimal assignment  $\mu^B$ , let  $\mathcal{S}_E(\mu^B) = \{s \in \mathcal{S} : \mu^B(s) \succ_s \mu^0(s) = \emptyset\} \subseteq \mathcal{S}$  be the set of students who *enter* the system, i.e., who are not initially assigned but are assigned to one of their preferences when capacities are expanded. Similarly, let  $\mathcal{S}_I(\mu^B) = \{s \in \mathcal{S} : \mu^B(s) \succ_s \mu^0(s) \succ_s \emptyset\} \subseteq \mathcal{S}$  be the set of students who *improve*, i.e., students who are initially assigned to some preference and improve their preference of assignment when capacities are expanded.<sup>6</sup>

In Theorem 1 we show that, when the penalty is small enough, the optimal solution will prioritize improvement by enhancing the assignment of initially assigned students.

<sup>6</sup> Note that the sets of students that *enter* and *improve* are match-specific. However, our results apply to any stable matching resulting from solving our problem.



**THEOREM 1.** *If  $r_{s,\emptyset} = 0$  for all  $s \in \mathcal{S}$  and  $|\{s \in \mathcal{S} : \exists c \in \mathcal{C} \text{ s.t. } c \succ_s \mu^0(s)\}| > B$  for  $B > 0$ , then  $|\mathcal{S}_I(\mu^B)| \geq |\mathcal{S}_E(\mu^B)|$ .*

On the other hand, if the penalty is sufficiently large, we show in Theorem 2 that the optimal solution will prioritize access by obtaining a stable matching with maximum cardinality.

**THEOREM 2.** *There is a sufficiently large  $r_{s,\emptyset}$  for all  $s \in \mathcal{S}$ , such that an optimal solution  $(\mathbf{x}^*, \mathbf{t}^*)$  of Problem 1 returns a maximum cardinality stable-matching.*

Note that Theorem 2 guarantees that the resulting stable matching will assign as many students as possible. However, it does not specify which students are prioritized. As we show in Proposition 1, we cannot guarantee that the number of students who enter will exceed those who improve.

**PROPOSITION 1.** *There exists an instance  $\Gamma$  of Problem 1 with  $B = 1$  for which there is no penalty  $r_{s,\emptyset}$  that guarantees  $|\mathcal{S}_E(\mu^B)| > |\mathcal{S}_I(\mu^B)|$ .*

All proofs can be found in Appendix A. As these results illustrate, the penalty values directly impact the outcome of Problem 1. In Section 5, we empirically analyze the trade-off between access and improvement using real data from the Chilean school choice system. For instance, we observe that if  $r_{s,\emptyset}$  is large, the optimal solution prioritizes initially unassigned students. On the other hand, if we set  $r_{s,\emptyset} = |\succ_s| + 1$ , we observe that the optimal solution considerably prioritizes the improvement of initially assigned students. Hence, which value of  $r_{s,\emptyset}$  to use is a policy-relevant question that impacts the trade-off between access and improvement.

### 2.3. Incentives

A mechanism is strategy-proof if truthful revelation is a dominant strategy for all agents. Roth (1982) and Dubins and Freedman (1981) prove that the student-proposing version of DA is strategy-proof for students. However, it is unclear whether this property holds when we include capacity decisions. For this reason, we now analyze the existence of a strategy-proof mechanism in the context of capacity expansion.

We say that agents (students or schools) have *ex-ante knowledge* about the capacity expansions if they know the expanded capacity of each school before applying. Similarly, we say that agents have *interim knowledge* if they know that the clearinghouse will add at least one seat, but they do not know the exact amount nor the allocation of those extra seats. Finally, we say that agents have *ex-post knowledge* if they know about capacity expansions only after they apply. Our first result, formalized in Theorem 3, shows that the timing in which the clearinghouse reveals the information about extra capacities plays an important role. Specifically, we prove that the mechanism is strategy-proof for students if they have ex-ante or ex-post knowledge about capacity expansions. However, this is not the case when students have interim knowledge.

**THEOREM 3.** *The mechanism is strategy-proof for students if they have ex-ante or ex-post knowledge about capacity expansions. In contrast, the algorithm is not strategy-proof for students if they have interim knowledge about capacity expansions.*

Theorem 3 shows that the timing of the disclosure of information about extra-capacities is critical to avoid manipulation from the students. Unfortunately, as we show in Theorem 4, the mechanism is not strategy-proof for schools in any case.

**THEOREM 4.** *The mechanism is not strategy-proof for schools, regardless of the timing by which the clearinghouse reveals the extra seats.*

## 2.4. Complexity

Our goal in this work is also to study different methodologies to solve Problem 1. Bobbio et al. (2022) analyze the complexity of this problem and prove that when the preference lists of the students are complete, i.e., they rank all the schools, Problem 1 is NP-hard. In this context, a possible approach is to design an approximation algorithm for this problem. Note that for a fixed  $\mathbf{t} \in \mathbb{Z}_+^c$ , the value

$$f(\mathbf{t}) := \min_{\mu} \left\{ \sum_{(s,c) \in \mu} r_{s,c} : \mu \text{ is a stable matching in instance } \Gamma_{\mathbf{t}} \right\} \quad (2)$$

can be computed in polynomial time by using the DA algorithm on instance  $\Gamma_{\mathbf{t}}$ . Therefore, one might be tempted to show that  $f$  is a lattice submodular function (Soma and Yoshida 2015).<sup>7</sup> However, this is not the case as we show in the following result.

**PROPOSITION 2.** *The function  $f(\mathbf{t})$  defined in Expression (2) is neither lattice submodular nor supermodular.*

We provide the corresponding counterexamples in Appendix A. Proposition 2 is in line with the results presented in (Bobbio et al. 2022), which assess the intractability of Problem 1.

The aforementioned hardness results motivate us to find exact formulations of the problem and to propose heuristics to obtain near-optimal solutions efficiently. We describe these in the next section.

## 3. Methodologies

In this section, we present different solution approaches. Specifically, in Section 3.1, we introduce two mathematical programs that out-of-the-box optimization solvers can solve to obtain an exact solution. Due to the potentially high computational time needed to solve these formulations, in Section 3.2 we describe two heuristics that can efficiently find good solutions.

<sup>7</sup> A function  $f : \mathbb{Z}_+^c \rightarrow \mathbb{R}_+$  is said to be lattice submodular if  $f(\mathbf{t} \vee \mathbf{t}') + f(\mathbf{t} \wedge \mathbf{t}') \leq f(\mathbf{t}) + f(\mathbf{t}')$  for any  $\mathbf{t}, \mathbf{t}' \in \mathbb{Z}_+^c$ , where  $\mathbf{t} \vee \mathbf{t}' := \max\{\mathbf{t}, \mathbf{t}'\}$  and  $\mathbf{t} \wedge \mathbf{t}' := \min\{\mathbf{t}, \mathbf{t}'\}$  component-wise. Function  $f$  is lattice supermodular if, and only if,  $-f$  is lattice submodular.

### 3.1. Integer Programming Formulations

Recalling that  $\mathbf{t}$  is the vector of extra seats allocated to the schools and  $B \in \mathbb{Z}^+$  is the total budget of additional seats that can be allocated, we define

$$\mathcal{P} = \left\{ (\mathbf{x}, \mathbf{t}) \in [0, 1]^\mathcal{E} \times [0, B]^\mathcal{C} : \sum_{c:(s,c) \in \mathcal{E}} x_{s,c} = 1 \text{ for all } s \in \mathcal{S}, \sum_{s \in \mathcal{S}} x_{s,c} \leq q_c + t_c \text{ for all } c \in \mathcal{C}, \sum_{c \in \mathcal{C}} t_c \leq B \right\},$$

representing the set of fractional (potentially non-stable) matchings with capacity expansion. Note that the first condition states that each student must be fully assigned to the schools in  $\mathcal{C} \cup \{\emptyset\}$ . The second condition sets a bound on the capacity of each school in  $\mathcal{C}$ , and the last condition establishes a budget for the extra seats to allocate. We denote by  $\mathcal{P}_\mathbb{Z}$  the integral points of  $\mathcal{P}$ , i.e.,  $\mathcal{P}_\mathbb{Z} = \mathcal{P} \cap (\{0, 1\}^\mathcal{E} \times \{0, \dots, B\}^\mathcal{C})$ .

Based on  $\mathcal{P}_\mathbb{Z}$ , we can formulate Problem 1 by adapting (1) to incorporate the decision vector  $\mathbf{t}$ . As a result, we obtain the following IQCP formulation:

$$\min_{\mathbf{x}, \mathbf{t}} \sum_{(s,c) \in \mathcal{E}} r_{s,c} \cdot x_{s,c} \tag{3a}$$

$$s.t. \quad (t_c + q_c) \cdot \left( 1 - \sum_{c' \succeq_s c} x_{s,c'} \right) \leq \sum_{s' \succ_c s} x_{s',c}, \quad \text{for all } (s, c) \in \mathcal{E}, \tag{3b}$$

$$(\mathbf{x}, \mathbf{t}) \in \mathcal{P}_\mathbb{Z}. \tag{3c}$$

Constraint (3b) guarantees that the matching is stable. Note that its relaxed version, i.e., when  $(\mathbf{x}, \mathbf{t})$  is allowed to be fractional, is a non-convex quadratic constraint, which makes the problem technically and computationally challenging. Finally, Constraint (3c) establishes the integrality of the decision variables and the matching feasibility.

In an attempt to obtain a mathematical programming model that can be solved efficiently in practice, we linearize the quadratic constraints (3b) through a McCormick envelope (McCormick 1976). In our formulation, the quadratic term  $t_c \cdot \sum_{c' \succeq_s c} x_{s,c'}$  in (3b) can be linearized in at least two ways. Specifically,

- *Aggregated Linearization*: For every  $(s, c) \in \mathcal{E}$ , we define

$$w_{s,c} := t_c \cdot \sum_{c' \succeq_s c} x_{s,c'},$$

- *Non-Aggregated Linearization*: For every  $(s, c) \in \mathcal{E}$  and  $c' \succeq_s c$ , we define

$$w_{s,c,c'} := t_c \cdot x_{s,c'}.$$

The mixed-integer programming formulation of the McCormick envelope for the aggregated linearization reads as

$$\min_{\mathbf{x}, \mathbf{t}, \mathbf{w}} \sum_{(s,c) \in \mathcal{E}} r_{s,c} \cdot x_{sc} \quad (4a)$$

$$s.t. \quad t_c - w_{s,c} + q_c \cdot \left( 1 - \sum_{c' \succeq_s c} x_{s,c'} \right) \leq \sum_{s' \succ_c s} x_{s',c}, \quad \text{for all } (s,c) \in \mathcal{E}, \quad (4b)$$

$$-w_{s,c} + t_c + B \cdot \sum_{c' \succeq_s c} x_{s,c'} \leq B, \quad \text{for all } (s,c) \in \mathcal{E}, \quad (4c)$$

$$w_{s,c} \leq t_c, \quad \text{for all } (s,c) \in \mathcal{E}, \quad (4d)$$

$$w_{s,c} \leq B \cdot \sum_{c' \succeq_s c} x_{s,c'}, \quad \text{for all } (s,c) \in \mathcal{E}, \quad (4e)$$

$$(\mathbf{x}, \mathbf{t}) \in \mathcal{P}_{\mathbb{Z}}, \quad \mathbf{w} \geq 0. \quad (4f)$$

Note that the original stability Constraints (3b) become linear Constraints (4b). Constraints (4c), (4d), (4e) and the non-negativity of the  $w_{s,c}$  form the McCormick envelope (see Appendix B for a short background). The rest of the constraints remain the same as well as the objective function.

It is well known that whenever at least one of the variables involved in the linearization is binary, the McCormick envelope is exact. Namely, the set of feasible solutions for the original problem coincides with the set of feasible solutions obtained from the McCormick linearization. For the aggregated linearization, we have that  $\sum_{c' \succeq_s c} x_{s,c'} \in \{0, 1\}$  due to matching constraints in  $\mathcal{P}_{\mathbb{Z}}$ .

**COROLLARY 1.** *The projection of the feasible region given by Constraints (4b)-(4f) in the variables  $\mathbf{t}$  and  $\mathbf{x}$  coincides with the region given by Constraints (3b) and (3c).*

We now discuss the mixed-integer programming formulation of the McCormick envelope for the non-aggregated linearization. Namely,

$$\min_{\mathbf{x}, \mathbf{t}, \mathbf{w}} \sum_{(s,c) \in \mathcal{E}} r_{s,c} \cdot x_{sc} \quad (5a)$$

$$s.t. \quad t_c - \sum_{c' \succeq_s c} w_{s,c,c'} + q_c \cdot \left( 1 - \sum_{c' \succeq_s c} x_{s,c'} \right) \leq \sum_{s' \succ_c s} x_{s',c}, \quad \text{for all } (s,c) \in \mathcal{E}, \quad (5b)$$

$$-w_{s,c,c'} + t_c + B \cdot x_{s,c'} \leq B, \quad \text{for all } (s,c) \in \mathcal{E}, \quad c' \succeq_s c, \quad (5c)$$

$$w_{s,c,c'} \leq t_c, \quad \text{for all } (s,c) \in \mathcal{E}, \quad c' \succeq_s c, \quad (5d)$$

$$w_{s,c,c'} \leq B \cdot x_{s,c'}, \quad \text{for all } (s,c) \in \mathcal{E}, \quad c' \succeq_s c, \quad (5e)$$

$$(\mathbf{x}, \mathbf{t}) \in \mathcal{P}_{\mathbb{Z}}, \quad \mathbf{w} \geq 0. \quad (5f)$$

In this case, Constraints (5c), (5d), (5e) and the non-negativity of the  $w_{s,c,c'}$  form the McCormick envelope. Similar to Corollary 1, since  $x_{s,c'} \in \{0, 1\}$ , we obtain the following corollary.

**COROLLARY 2.** *The projection of the feasible region given by Constraints (5b)-(5f) in the variables  $\mathbf{t}$  and  $\mathbf{x}$  coincides with the region given by Constraints (3b) and (3c).*

Therefore, both mixed-integer linear programming formulations yield the same set of feasible solutions. Interestingly, the feasible region of the relaxed aggregated linearization, i.e., when  $(\mathbf{x}, \mathbf{t}) \in \mathcal{P}_{\mathbb{Z}}$  in (4f) is changed to  $(\mathbf{x}, \mathbf{t}) \in \mathcal{P}$ , is strictly contained in the feasible region of the relaxed non-aggregated linearization.

**THEOREM 5.** *The feasible region of the relaxed aggregated linearization model is contained in the feasible region of the relaxed non-aggregated linearization model.*

*Proof.* The constraints that do not involve the linearization terms  $w_{s,c}$  or  $w_{s,c,c'}$  are trivially satisfied by a feasible solution in both formulations. Therefore, we will restrict our analysis to the remaining constraints. Let  $(\mathbf{x}, \mathbf{t}, \mathbf{w})$  be a feasible solution of the relaxed aggregated linearized program. It is easy to verify that by defining  $\bar{w}_{s,c,c'} = w_{s,c} \cdot x_{s,c'}$  for every  $s \in \mathcal{S}$ ,  $c \in \mathcal{C}$  and  $c' \succeq_s c$ , the constraints of the relaxed non-aggregated linearization are all met.  $\square$

Theorem 5 implies that the optimal value of the relaxed aggregated linearized model is greater than or equal to the optimal value of the relaxed non-aggregated linearized model. In Example 2 in Appendix A, we provide a counterexample that shows that the inverse of Theorem 5 is not true, i.e., the inclusion in Theorem 5 is strict. Since solution approaches to mixed-integer programming formulations are based on the quality of their continuous relaxation, the aggregated linearization dominates the non-aggregated one, thus it is expected to perform better in practice.

### 3.2. Heuristics

In this section, we present two natural methods: (i) a greedy approach (Greedy), and (ii) an LP-based heuristic (LPH). These two heuristics rely on the computation of a student-optimal stable matching, which can be done in polynomial time using the DA algorithm. For completeness, we include a description of DA in Appendix B.1.

*Greedy Approach.* In Greedy, we explore the fact that the objective function is decreasing in  $\mathbf{t}$  and iteratively assign an extra seat to the school leading to the largest reduction in the objective. More precisely, Greedy performs  $B$  sequential iterations. In each iteration, we evaluate the objective function for each possible allocation of one extra seat using DA. Then, the school leading to the lowest objective receives that extra seat. At the end of this procedure,  $B$  extra seats are allocated. In Algorithm 1, we formalize our Greedy heuristic.

In the algorithm above,  $\mathbf{1}_c \in \{0, 1\}^{\mathcal{C}}$  denotes the indicator vector whose value is 1 in component  $c \in \mathcal{C}$  and 0 otherwise. Recall that, for a given  $\mathbf{t}$ ,  $f(\cdot)$  can be evaluated in polynomial time using the DA algorithm.

---

**Algorithm 1** Greedy

---

**Input:** An instance  $\Gamma = \langle \mathcal{S}, \mathcal{C}, \succ, \mathbf{q} \rangle$  and a non-negative integer budget  $B$ .**Output:** A feasible allocation  $\mathbf{t}$  and a stable matching  $\mu$  in the expanded instance  $\Gamma_{\mathbf{t}}$ .

- 1: Initialize  $\mathbf{t} = \mathbf{0}$  the zero vector.
  - 2: **while**  $\sum_{c \in \mathcal{C}} t_c < B$  **do**
  - 3:      $c^* \in \operatorname{argmax} \{f(\mathbf{t} + \mathbf{1}_c) : c \in \mathcal{C}\}$ , where  $f$  is defined as in Expression (2).
  - 4:      $\mathbf{t} \leftarrow \mathbf{t} + \mathbf{1}_{c^*}$ .
- 

*LP-based Heuristic.* If we relax the stability constraints, Problem 1 can be formulated as a minimum-cost flow problem whose polytope has integer vertices. Once we enrich this problem with the expansion of capacities, the integrality of the vertices is preserved. Hence, LPH starts by solving the linear program that minimizes Objective (3a), restricted to the set  $\mathcal{P}$ . As a result, we obtain an allocation of extra seats  $\mathbf{t}^*$  and an assignment  $\mathbf{x}^*$  that is not necessarily stable—recall that  $\mathcal{P}$  is the space of fractional (potentially non-stable) matchings. Then, using the DA algorithm, we compute the student-optimal stable matching in the new instance that considers the capacity expansion obtained by the linear program. In Algorithm 2, we formalize our LPH heuristic.

---

**Algorithm 2** LPH

---

**Input:** An instance  $\Gamma = \langle \mathcal{S}, \mathcal{C}, \succ, \mathbf{q} \rangle$  and a non-negative integer budget  $B$ .**Output:** A feasible allocation  $\mathbf{t}$  and a stable matching  $\mu$  in the expanded instance  $\Gamma_{\mathbf{t}}$ .

- 1: Obtain  $(\mathbf{x}^*, \mathbf{t}^*) \in \operatorname{argmax} \left\{ \sum_{(s,c) \in \mathcal{E}} r_{s,c} \cdot x_{s,c} : (\mathbf{x}, \mathbf{t}) \in \mathcal{P} \right\}$ .
  - 2: Compute stable matching  $\mu$  in instance  $\Gamma_{\mathbf{t}^*}$  using the DA algorithm.
- 

## 4. Evaluation of Methods on Random Instances

In this section, we empirically evaluate the performance of our methods to assess which formulations and heuristics work better. To perform this analysis, we assume that students have complete preference lists and that the sum of schools' capacities equals the number of students. Since the number of variables and constraints increases with  $|\mathcal{E}|$ , considering complete preference lists increases the dimension of the problem and thus makes it harder to solve, providing a “worst-case scenario” in terms of computing time.

*Experimental Setup.* We create 30 instances for each combination of the following parameters:  $|\mathcal{S}| \in \{1000, 2000\}$ ,  $|\mathcal{C}| \in \{5, 8, 10, 15\}$ ,  $B \in \{1, 2, 5, 10, 20, 30\}$ . In particular, for each instance, we generate preference lists and capacities uniformly at random, ensuring that no school has zero capacity. The formulations were coded in Python 3.7.3 and solved through Gurobi 9.1.2, restricted

$ \mathcal{S} $	$ \mathcal{C} $	Integer Quadratic Program		Aggregated Linearization	
		rows	columns	rows	columns
1000	5	1006	5005	21006	10005
	8	1009	8008	33009	16008
	10	1011	10010	41011	20010
	15	1016	15015	61016	30015
2000	5	2006	10005	42006	20005
	8	2009	16008	66009	32008
	10	2011	20010	82011	40010
	15	2016	30015	122016	60015

Table 1 Size of the formulations.

to a single CPU thread and 1 hour of time limit. The scripts were run on an Intel(R) Xeon(R) Gold 6226 CPU on 2.70GHz, running Linux 7.9.<sup>8</sup> In Table 1, we summarize the initial size of our mathematical programming formulations, namely, the number of constraints (rows) and variables (columns) at the beginning of the computation. Note that the models' size does not depend on the budget for extra seats and only on the number of students and schools. Also, note that the IQCP formulation is more compact than the aggregated linearization model for the same set of parameters  $|\mathcal{S}|$  and  $|\mathcal{C}|$ .

*Experiments.* In Table 3 (in Appendix D), we present the results of our experiments, where we compare the quadratic program (IQP), the aggregated linearization (Agg-Lin), and the two heuristics (Greedy and LPH).

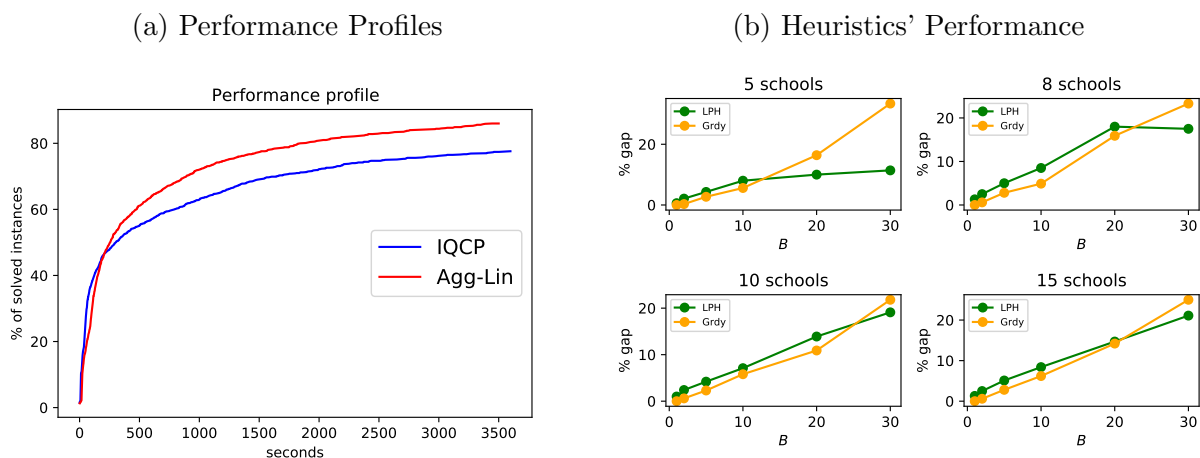


Figure 1 Figure on the left: Performance profiles of the integer quadratic program and the aggregated linearization (experiments on 1440 instances). Figure on the right: The average gap (in %) between the two heuristics and the best upper bound found by the exact methods; the gap is represented as a function of  $B$ . The number of students is 2000 and each data-point is the average over 30 instances.

<sup>8</sup>The code and the synthetic instances are publicly available on GitHub at <https://github.com/federicobobbio/Capacity-expansion-in-the-college-admission-problem>.

First, we observe that Agg-Lin outperforms IQP starting in  $|\mathcal{S}| \geq 1000$ ,  $|\mathcal{C}| = 15$  and  $B \geq 1$ . Moreover, given that the difference in performance is negligible in other smaller instances, we conclude that Agg-Lin should be preferred. This result is in line with Figure 1a, which shows the fraction of instances solved to optimality (given a mipgap tolerance of  $1e^{-4}$ ) within one hour by each method. Second, by looking at the computing times and gaps for the exact methods, we observe that the main driver of the computing time is the number of schools. Finally, we find that increasing  $B$  reduces the computing time of the exact models. In particular, we observe a drastic reduction in the computing time as we move from  $B = 1$  to  $B = 2$ . One possible explanation is that the additional extra seat makes infeasible the branching in the second summation of the stability constraint.

Regarding our heuristics, we find that Greedy performs better than LPH when  $B$  is relatively small, but this result reverses as  $B$  increases. This result is in line with Figure 1b, where we show the average gap between the heuristics and the best upper bound considering  $|\mathcal{S}| = 2000$  and  $|\mathcal{C}| \in \{5, 8, 10, 15\}$  (the results are similar with  $|\mathcal{S}| = 1000$ ).

## 5. Application to School Choice in Chile

To illustrate the potential benefits of capacity expansion, we adapt our framework to the Chilean school choice system. This system, introduced in 2016 in the southernmost region of the country (Magallanes), was fully implemented in 2020 and serves close to half a million students and more than eight thousand schools each year.

The Chilean school choice system is a good application for our methodology for multiple reasons. First, the system uses a variant of the student-proposing Deferred Acceptance algorithm, which incorporates priorities and overlapping quotas. Our framework can include all the features of the Chilean system, including the block application, the dynamic siblings' priority, etc. We refer to Correa et al. (2022) for a detailed description of the Chilean school choice system and the algorithm used to perform the allocation. Second, the Ministry of Education manages all schools that participate in the system and thus can ask them to modify their vacancies within a reasonable range. Finally, the system is currently being redesigned, and we are collaborating with the authorities to include some of the ideas introduced in our work.

### 5.1. Data and Simulation Setting

We consider data from the admission process in 2018.<sup>9</sup> Specifically, we focus on the southernmost region of the country as it is the region where all policy changes are first evaluated. Moreover, we restrict the analysis to Pre-K for two reasons: (1) it is the level with the highest number of

<sup>9</sup> All the data is publicly available here.



**Table 2 Instance for Evaluation**

Region	Students	Schools	Applications
Magallanes Pre-K	1389	43	4483
Overall Pre-K	84626	3465	256120
Overall (all levels)	274990	6421	874565

applicants, as it is the first entry level in the system, and (2) to speed up computation. In Table 2, we report summary statistics about the instance, and we compare it with the values nationwide for the same year.

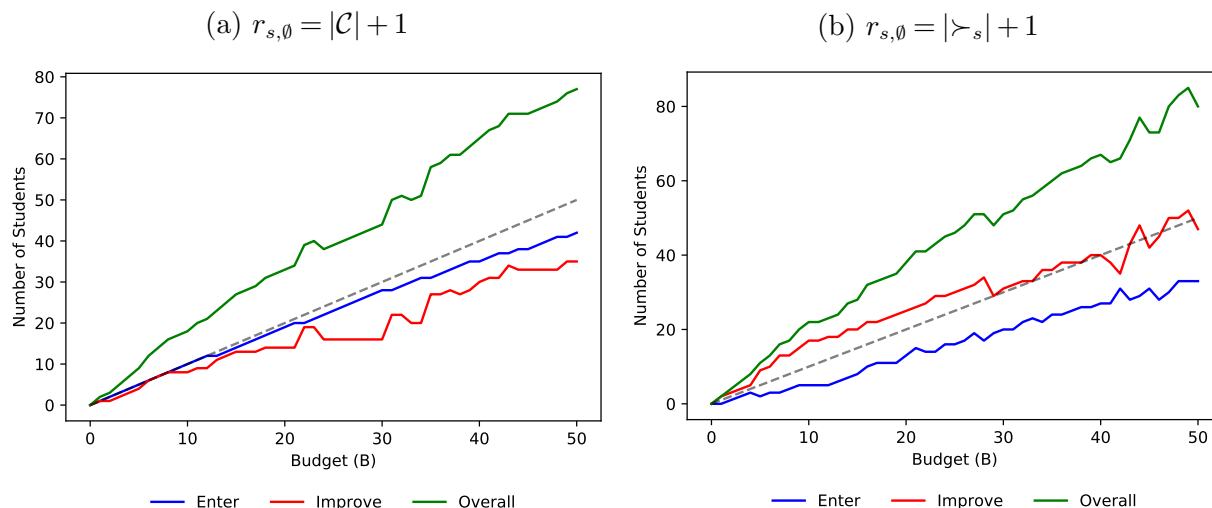
We perform our simulations varying the budget  $B \in \{0, 1, \dots, 50\}$  and the penalty for unassigned students  $r_{s,\emptyset}$ . For the latter, we consider two cases: (i)  $r_{s,\emptyset} = |\mathcal{C}| + 1$  for all  $s \in \mathcal{S}$ , and (ii)  $r_{s,\emptyset} = |\succ_s| + 1$  for all  $s \in \mathcal{S}$ . Notice that the two values for  $r_{s,\emptyset}$  cover two extreme cases. When  $r_{s,\emptyset} = |\mathcal{C}| + 1$  (or any large number), the model will use the extra vacancies to ensure that a student that was previously unassigned gets assigned. In contrast, when  $r_{s,\emptyset} = |\succ_s| + 1$ , the model will (most likely) assign the extra seat to the school that leads to the largest chain of improvements. Hence, from a practical standpoint, which penalty to use is a policy-relevant decision that must balance *access* and *improvement*.

## 5.2. Results

We report our main simulation results in Figure 2. For each budget, we plot the number of students who (1) *enter* the system, i.e., who are not initially assigned (with  $B = 0$ ), but are assigned to one of their preferences when capacities are expanded; (2) *improve*, i.e., students who are initially assigned to some preference but improve their preference of assignment when capacities are expanded; and (3) *overall*, which is the total number of students who benefit relative to the baseline and is equal to the sum of the number of students who *enter* and *improve*.<sup>10</sup>

First, we confirm that all initially assigned students (with  $B = 0$ ) get a school at least as preferred when we expand capacities. Second, increasing capacities with a high penalty primarily benefits initially unassigned students. In contrast, students who improve their assignments are the ones who most benefit when the penalty is low. Third, we observe that the total number of students who benefit (in green) is considerably larger than the number of additional seats (dashed). The reason is that an extra seat can lead to a chain of improvements that ends either on a student that *enters* the system or in a school that is under-demanded. Finally, we observe that the total number of students who gain from the additional seats is not strictly increasing in the budget. Indeed, the number may decrease if the extra seat allows a student to dramatically enhance their

<sup>10</sup> For the case with  $r_{s,\emptyset} = |\succ_s| + 1$ , we consider a MipGap tolerance of 0.1% starting from  $B = 35$ . The reason is that the computational time increases considerably for larger budgets.

**Figure 2** Effect for Students

assignment (e.g., moving from being unassigned to assigned to their top preference). This effect on the objective could be larger than that of a chain of minor improvements involving several students, and thus the number of students who benefit may decrease.

In Figure 3, we analyze the impact of expanding capacities on the number of schools with increased capacity and the maximum number of additional seats per school. We observe that the number of schools with extra seats remains relatively stable as we increase the budget. In addition, we observe that the maximum number of additional seats in a given school increases with the budget. This is because students' preferences are highly correlated (i.e., students have similar preferences), and thus, a few over-demanded schools concentrate the extra seats added to the system. Finally, the latter effect is more prominent when the penalty is lower.

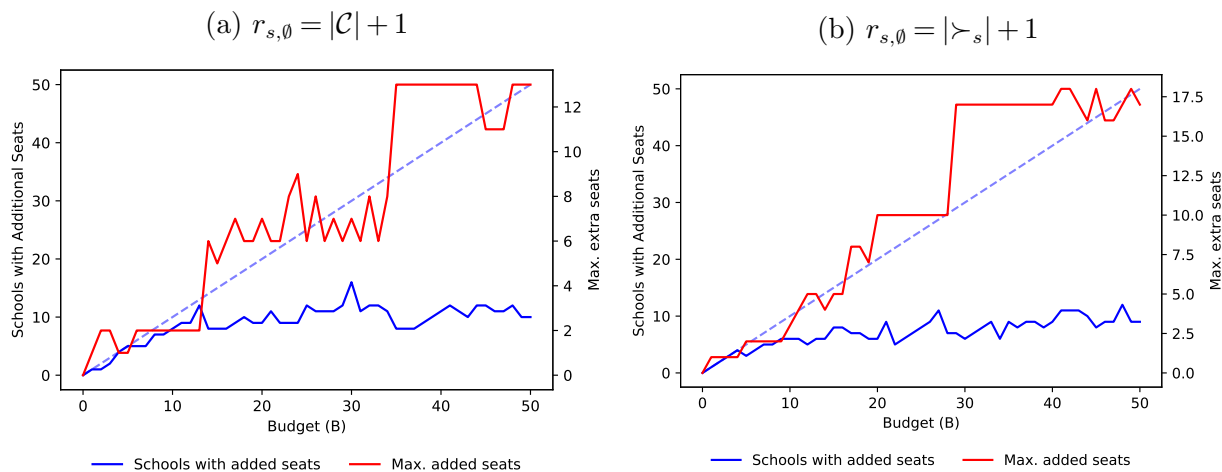
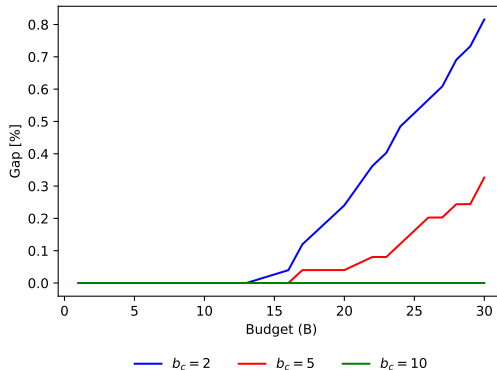
**Figure 3** Effect for Schools

Figure 4 Effect of Bounds on Expansion



**5.2.1. Practical Implementation.** A valid concern from policy-makers is that our approach would assign most extra seats to a small number of over-demanded schools. As a result, our solution would not be feasible in practice since classroom capacities limit the possible expansion. To rule out this concern, we adapt our model and include the new set of constraints

$$t_c \leq b_c, \quad \text{for all } c \in \mathcal{C},$$

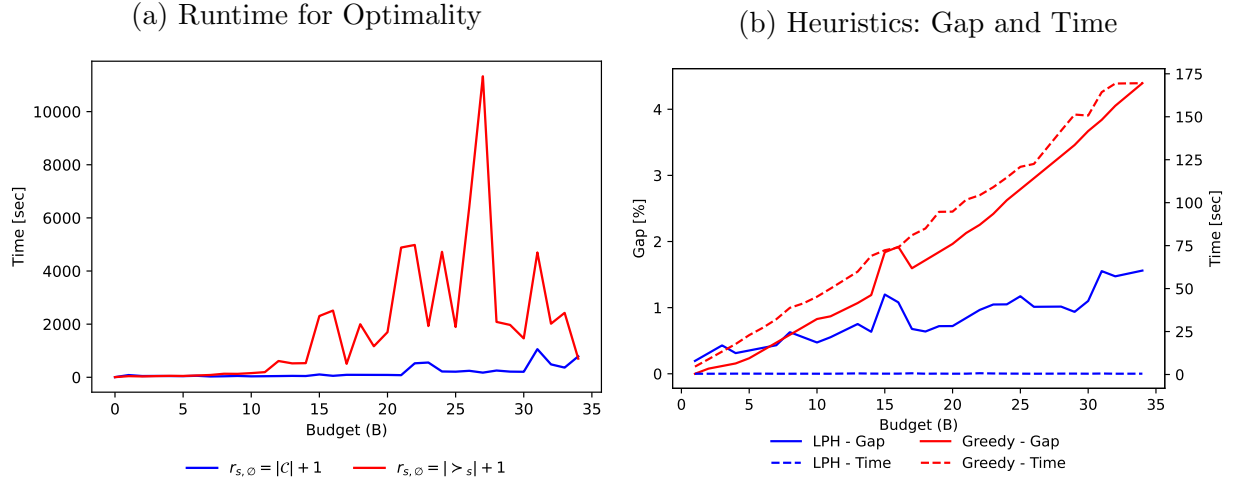
where  $b_c$  is the maximum number of additional seats that can be allocated to school  $c$ .

In Figure 4, we compare the gap between the optimal (unconstrained) solution and the values obtained when considering  $b_c \in \{2, 5, 10\}$  for all  $c \in \mathcal{C}$ . First, we observe that the gap increases for  $b_c \in \{2, 5\}$  as we increase the budget, while it does not change for  $b_c = 10$ . This result suggests that the problem has many optimal solutions, and thus we can select one that does not over-expand some schools. Second, we observe that the overall gap is relatively low (max of 0.8%), which suggests that we can include the practical limitations for schools without major loss of performance. In Appendix C, we discuss some model extensions to incorporate other relevant aspects from a practical standpoint.

**5.2.2. Heuristics.** In Figure 5a, we plot the time to achieve zero optimality gap for  $B \in \{0, \dots, 35\}$ . We observe that the computational time for the case with  $r_{s,\emptyset} = |\succ_s| + 1$  is considerably larger than that for  $r_{s,\emptyset} = |\mathcal{C}| + 1$ . Moreover, we observe that the solution time for  $r_{s,\emptyset} = |\succ_s| + 1$  can be considerably large for some budget values, and this problem may be exacerbated if we consider larger instances (e.g., larger regions or the entire country). For this reason, it is relevant to study the performance of the heuristics described in Section 3.2.

For each value of the budget, in Figure 5b we report the gap obtained relative to the optimal policy<sup>11</sup> and the execution time of each heuristic. As shown in our simulation results, Greedy

<sup>11</sup> We consider the problem with penalty  $r_{s,\emptyset} = |\succ_s| + 1$ . The results are similar if we consider  $r_{s,\emptyset} = |\mathcal{C}| + 1$

**Figure 5** Heuristics

performs better than LPH for low values of the budget, but this reverses as the budget increases. In addition, we observe that the time of LPH is significantly better than that of Greedy. Hence, we conclude that LPH can be an effective approach for large instances and large values of  $B$ .

## 6. Conclusions

We studied how centralized clearinghouses can jointly decide how to allocate additional seats and find a stable matching. To accomplish this: we introduced a model of stable matching under capacity planning, and we provided mathematical programming formulations to solve the problem. Given the complexity of the problem Bobbio et al. (2022), we proposed two heuristics that find close-to-optimal solutions efficiently. Finally, we adapted our framework to solve an instance of the Chilean school choice problem.

Our results show that each additional seat can benefit multiple students. However, depending on how we penalize having unassigned students in the objective, the set of students who benefit from the extra seats changes. Indeed, we have shown that if we consider a large penalty, the optimal solution prioritizes *access*, i.e., assigning students that were previously unassigned. In contrast, if that penalty is low, the optimal solution prioritizes *improvement*, i.e., benefiting as much as possible students' preference of assignment. Hence, which penalty to consider is a policy-relevant decision that depends on the objective of the clearinghouse.

Our work opens the door to several future research directions. First, we are currently working on designing methods to find exact solutions in large instances of the problem. Second, it would be interesting to understand how students' incentives change when they know that the clearinghouse may expand its capacities. Finally, understanding how to set the penalties to achieve an intermediate balance between *access* and *improvement* is also an exciting direction for future work.

## Appendix

To ease the exposition, we avoid using the symbol  $\succ$  when presenting a preference list, instead we simply separate agents by “,” and use the convention that the leftmost agents are the most preferred. For instance, we will represent the preference list  $w \succ w' \succ w''$  as  $w, w', w''$ . The “...” at the end of a preference list will be used to represent an arbitrary completion of the preferences.

### Appendix A: Proofs

*Proof of Lemma 1.* We begin by noting that Baiou and Balinski (2000) show that the feasible region of the Integer Program (1) corresponds to the set of stable matchings. Therefore, in the following proof, we only need to focus on proving the equivalence between the student-optimal matching and a matching minimizing the sum of the student’s rank over the set of stable matchings.

**If:** In a student-optimal stable matching, each student is assigned to the best school they could achieve in any stable matching Gale and Shapley (1962). Thus, each unassigned student is unassigned in every stable matching. Moreover, by the Rural Hospital Theorem, the same students are assigned in all stable matchings. Suppose that  $\mu$  is the student-optimal stable matching but its corresponding binary encoding is not optimal to the Integer Program (1). Let  $x'$  be an optimal solution to the Integer Program (1) and let  $\mu' = \{(s, c) \in \mathcal{E} : x'_{s,c} = 1\}$  be the associated matching. Hence, the following inequality holds:  $\sum_{(s,c) \in \mu} r_{s,c} > \sum_{(s,c) \in \mu'} r_{s,c}$ . This means that there is at least one student  $s'$  who prefers the stable matching  $\mu'$  to the stable matching  $\mu$ , which is a contradiction.

**Only if:** Let  $\mu$  be a (stable) matching corresponding to an optimal solution of the Integer Program (1). As before, by the Rural Hospital Theorem, we observe that the set of students unassigned in  $\mu$  is the same set of students unassigned in every stable matching. Hence, the set of assigned students in  $\mu$ , is the same set for every stable matching. Let us suppose, again by contradiction, that  $\mu$  is not a student-optimal stable matching. Let  $\mu'$  be a student-optimal stable matching. Denote by  $S'$  the set of students whose assignment to schools differs in the two matchings. By construction, the objective value of the Integer Program (1) for  $S \setminus S'$  is the same in both  $\mu$  and  $\mu'$ :

$$\sum_{(s,c) \in \mu: s \in S \setminus S'} r_{s,c} = \sum_{(s,c) \in \mu': s \in S \setminus S'} r_{s,c}.$$

Furthermore,  $S'$  is the disjoint union of the following two sets of students:  $S'_1$ , the set of assigned students who prefer their school in  $\mu'$ , and  $S'_2$ , the set of assigned students who prefer their school in  $\mu$ . By Gale and Shapley (1962),  $\mu'$  is a stable matching in which all students are assigned the best school they could achieve in any stable matching. Therefore, the set  $S'_2$  is empty. Hence,  $S' = S'_1$ , and by hypothesis

$$\sum_{(s,c) \in \mu: s \in S'_1} r_{s,c} < \sum_{(s,c) \in \mu': s \in S'_1} r_{s,c}.$$

However, from the definition of  $S'_1$ ,  $r_{s,\mu'(s)} \leq r_{s,\mu(s)}$  for every  $s \in S'_1$ , which leads to a contradiction.  $\square$

EXAMPLE 1. Consider an instance with three schools  $\mathcal{C} = \{c_1, c_2, c_3\}$ , four students  $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$ , and capacities  $q_{c_1} = q_{c_2} = 1$ ,  $q_{c_3} = 2$ . In addition, consider preferences given by:

$$c : s_1 \succ s_2 \succ s_3 \succ s_4, \forall c \in \mathcal{C}$$

$$s_1 : c_1 \succ c_2 \succ c_3$$

$$s_2 : c_2 \succ c_1 \succ c_3$$

$$s_3 : c_1 \succ c_3 \succ c_2$$

$$s_4 : c_2 \succ c_3 \succ c_1.$$

Notice that, with no capacity expansion, the student-optimal stable matching is

$$\mu^* = \{(s_1, c_1), (s_2, c_2), (s_3, c_3), (s_4, c_3)\},$$

which leads to an objective of 6. On the other hand, if we have a budget  $B = 1$ , note that we can allocate it to either  $c_1$  and obtain the matching  $\mu' = \{(s_1, c_1), (s_2, c_2), (s_3, c_1), (s_4, c_3)\}$ , or to school  $c_2$  and obtain the matching  $\mu'' = \{(s_1, c_1), (s_2, c_2), (s_3, c_3), (s_4, c_2)\}$ . In both cases, one student moves from their second choice to their top choice, and thus in both cases the sum of preferences of assignment is 5. Hence, we conclude that this problem has more than one optimal solution.

We now focus on the proof of Theorem 1. Towards this goal, we first prove some lemmas. Recall that the student-optimal DA algorithm (see Appendix B.1) does not depend on the order that the students apply to schools. For the remainder of this section, consider a fix ordering  $s_1, \dots, s_{|\mathcal{S}|}$ . Now, we analyze the effect of an additional seat to the assignment of the students. Given an additional seat in some school, we define the *chain of improvements* as the sequence of students who change their assignment (i.e., either improve or enter) with respect to the instance without the additional seat. This sequence of students is determined by the ordering in which the students apply in the DA algorithm. Note that this sequence is optimal for the students since we are considering the student-optimal DA method. Also, observe that an additional seat in a different school may lead to a different chain of improvements. Before proving Theorem 1, our goal is to show some properties of these chains of improvements.

Recall that a school  $c \in \mathcal{C}$  is said to be under-demanded in an assignment  $\mu$  if there exists no student  $s'$  such that  $c \succ_{s'} \mu(s')$ . A school is over-demanded if it is not under-demanded.

LEMMA 2. *Any chain of improvements ends in a student that was initially unassigned or in a student who was previously assigned to an under-demanded school.*

*Proof.* Denote by  $\mu$  the initial matching for  $B = 0$ . Suppose that the lemma is not true, i.e., there exists a chain of improvements that ends in a student  $s \in \mathcal{S}$  that was assigned in  $\mu$  to an over-demanded school  $c \in \mathcal{C}$ . Since  $s$  is part of the chain of improvements, we know that  $s$  is assigned to a preferred school  $c' \succ_s c$  in the new assignment. Given that  $c$  was over-demanded in  $\mu$ , we know there exists a student  $s' \in \mathcal{S}$  such that  $c \succ_{s'} \mu(s')$  and the seat that was previously used by  $s$  in  $c$  is available. Thus, student  $s'$  would be part of the chain of improvements. This is a contradiction, since we assumed that the chain ended with student  $s$ .  $\square$

LEMMA 3. *Any chain of improvements reaches at most one initially unassigned student.*

*Proof.* Since every student in a chain of improvements uses only one seat in some school, we know the seat that the student frees can be used by only one student, who in turn liberates only one seat, and so on. Then, by Lemma 2, we know that the chain continues the same way until it reaches an under-demanded school or an initially unassigned student. In both cases, the chain ends in at most one previously unassigned student.  $\square$

Now, our goal is to prove that  $B$  additional seats lead to at most  $B$  chains of improvements. Given a budget  $B > 0$ , we denote by  $(c_1, \dots, c_B)$  the sequence of additional seats (possibly with repetition) where in each step  $\ell \in \{1, \dots, B\}$  we allocate an additional seat to school  $c_\ell \in \mathcal{C}$  and then we use the DA algorithm to find the student-optimal stable matching considering that extra seat.

LEMMA 4. *Fix a budget  $B > 0$ . Any permutation of a given sequence of additional seats  $(c_1, \dots, c_B)$  leads to the same final matching. Moreover, a sequence of  $B$  additional seats generates at most  $B$  chains of improvements.*

*Proof.* Note that any permutation of a sequence of additional seats  $(c_1, \dots, c_B)$  leads to the same allocation vector  $\mathbf{t} \in \mathbb{Z}_+^{\mathcal{C}}$  with  $t_c = |\{c' \in (c_1, \dots, c_B) : c' = c\}|$ . Since the outcome of the DA algorithm is independent on the order in which students are processed, then it is irrelevant the order in which the sequence of additional seats is processed. In fact, this is equivalent to using the DA algorithm in the expanded instance  $\Gamma_{\mathbf{t}}$ .

Let us prove now the second part of the lemma. By using Lemmas 2 and 3 we know that a chain of improvements starts with a student and ends in either an under-demanded school or an initially unassigned student. Then, each additional seat in a sequence  $(c_1, \dots, c_B)$  generates a single chain of improvements, and since we know that at most  $B$  additional seats will be allocated, we know that there will be at most  $B$  chains of improvements.  $\square$

*Proof of Theorem 1.* For the purpose of the proof, let us consider an arbitrary optimal allocation  $\mathbf{t}^*$ . Let  $(c_1^*, \dots, c_B^*)$  be any sequence of additional seats defined by  $\mathbf{t}^*$ . First, note that a chain of improvements will never start in a student  $s$  that was initially unassigned because  $r_{s, \emptyset} < r_{s, c}$  for all  $c \neq \emptyset$ . Hence, each additional seat  $c_\ell^*$  generates a chain of improvements that starts with an initially assigned student. In addition, from Lemma 3 we know that each additional seat  $c_\ell^*$  creates a chain that reaches at most one initially unassigned student. These two observations above imply that each additional seat generates a chain in which the number of students that improve in that chain is at least the number of students who enter in that chain. Finally, from Lemma 4 we know that  $(c_1^*, \dots, c_B^*)$  generates at most  $B$  chains of improvements. Thus, by summing over all the chains the number students who enter and improve, we conclude that  $|\mathcal{S}_I(\mu^B)| \geq |\mathcal{S}_E(\mu^B)|$ .  $\square$

*Proof of Theorem 2.* Let  $\mu^*$  be the stable-matching corresponding to the solution  $(\mathbf{x}^*, \mathbf{t}^*)$ . To find a contradiction, suppose there exists another stable matching  $\mu'$  that has a higher cardinality, i.e.,  $|s \in \mathcal{S} : \mu'(s) = \emptyset| < |s \in \mathcal{S} : \mu^*(s) = \emptyset|$ .

By optimality of  $(\mathbf{x}^*, \mathbf{t}^*)$ , we know that

$$\sum_{s \in \mathcal{S}} r_{s, \mu^*(s)} < \sum_{s \in \mathcal{S}} r_{s, \mu'(s)}.$$

On the other hand, we know that

$$\begin{aligned}
\sum_{s \in \mathcal{S}} r_{s, \mu^*(s)} - \sum_{s \in \mathcal{S}} r_{s, \mu'(s)} &= \sum_{s \in \mathcal{S}: \mu^*(s) \in \mathcal{C}} r_{s, \mu^*(s)} - \sum_{s \in \mathcal{S}: \mu'(s) \in \mathcal{C}} r_{s, \mu'(s)} + \sum_{s \in \mathcal{S}: \mu^*(s) = \emptyset} r_{s, \emptyset} - \sum_{s \in \mathcal{S}: \mu'(s) = \emptyset} r_{s, \emptyset} \\
&= \sum_{s \in \mathcal{S}: \mu^*(s) \in \mathcal{C}} r_{s, \mu^*(s)} - \sum_{s \in \mathcal{S}: \mu'(s) \in \mathcal{C}} r_{s, \mu'(s)} \\
&\quad + \bar{r} \cdot [|\mathcal{S} : \mu^*(s) = \emptyset| - |\mathcal{S} : \mu'(s) = \emptyset|] \\
&> - \sum_{s \in \mathcal{S}} |\gamma_s| + \bar{r} \cdot [|\mathcal{S} : \mu^*(s) = \emptyset| - |\mathcal{S} : \mu'(s) = \emptyset|] \\
&> 0.
\end{aligned} \tag{6}$$

The first equality follows from assuming that  $r_{s, \emptyset} = \bar{r}$  for all  $s \in \mathcal{S}$ . The first inequality follows from the fact that, given a student  $s$  that is assigned, the maximum improvement is to move from their last preference,  $|\gamma_s|$ , to their top preference, and therefore  $r_{s, \mu^*(s)} - r_{s, \mu'(s)} > 1 - |\gamma_s| > -|\gamma_s|$ . Then, we have that  $\sum_{s \in \mathcal{S}: \mu^*(s) \in \mathcal{C}} r_{s, \mu^*(s)} - \sum_{s \in \mathcal{S}: \mu'(s) \in \mathcal{C}} r_{s, \mu'(s)} \geq - \sum_{s \in \mathcal{S}} |\gamma_s|$ . Finally, the last inequality follows from the fact that

$$|\mathcal{S} : \mu^*(s) = \emptyset| - |\mathcal{S} : \mu'(s) = \emptyset| \geq 1$$

and that  $\bar{r}$  is arbitrarily large, and thus we can take a value  $\bar{r} > \sum_{s \in \mathcal{S}} |\gamma_s|$ . As a result, we obtain that

$$\sum_{s \in \mathcal{S}} r_{s, \mu^*(s)} - \sum_{s \in \mathcal{S}} r_{s, \mu'(s)} > 0,$$

which contradicts the optimality of  $(\mathbf{x}^*, \mathbf{t}^*)$ .  $\square$

*Proof of Proposition 1.* Consider  $\mathcal{S} = \{s_1, s_2, s_3\}$  and  $\mathcal{C} = \{c_1, c_2\}$ , with  $q_c = 1$  for all  $c$ . In addition, consider the following preferences and priorities:  $s_1, s_3 : c_1$ ,  $s_2 : c_1, c_2$ ,  $c_1 : s_1, s_2, s_3$  and  $c_2 : s_2, s_1, s_3$ . Then, if  $B = 1$ , the extra seat will go to  $c_1$ , and  $s_3$  will remain unassigned.  $\square$

*Proof of Theorem 3.* If students have ex-ante or ex-post knowledge about capacity expansions, then it is direct that the mechanism is strategy-proof for students. The reason is that, in the ex-ante and ex-post cases, capacities are fixed (from the perspective of students), and thus the problem is equivalent to the standard setting, in which we know that student-proposing DA is strategy-proof for students. Then, since our mechanism finds the student-optimal stable-matching, we conclude that our mechanism is strategy-proof for students following the same argument in Roth (1982), Dubins and Freedman (1981).

For the second part, we consider the following counter-example:

$$\begin{array}{ll}
s_1 : c_1 \succ \dots & c_1 : s_1 \succ s_3 \succ \dots \\
s_2 : c_2 \succ \dots & c_2 : s_2 \succ s_3 \succ \dots \\
s_3 : c_1 \succ c_2 \succ c_3 \succ \dots & c_3 : s_3 \succ \dots \\
s'_1 : c'_1 \succ \dots & c'_1 : s'_1 \succ s'_2 \succ \dots \\
s'_2 : c'_1 \succ c'_2 \succ \dots & c'_2 : s'_2 \succ \dots
\end{array}$$

where the “...”, represent an arbitrary completion of the preferences. If agents are truthful, the optimal allocation is to assign the extra seat to school  $c_1$ , which will admit student  $s_3$ ; thus, the final matching would be  $\{(s_1, c_1), (s_2, c_2), (s_3, c_1), (s'_1, c'_1), (s'_2, c'_2)\}$ . If student  $s'_2$  misreports her preferences by reporting

$$s'_2 : c'_1 \succ c_1 \succ c_2 \succ c'_2 \succ c_3,$$

the extra seat would be allocated to  $c'_1$  and  $s'_2$  would get her favorite school.  $\square$



*Proof of Theorem 4.* If schools have ex-ante or ex-post knowledge about capacity expansions, then it is direct that the mechanism is not strategy-proof for schools. The reason is that, in the ex-ante and ex-post cases, capacities are fixed (from the perspective of schools), and thus the problem is equivalent to the standard setting, in which we know that student-proposing DA is not strategy-proof for schools Sönmez (1997).

If schools have interim knowledge of the extra spots, then consider the following example consisting of three schools  $c_1, c_2, c_3$  all with capacity 2, and three students  $s_1, s_2, s_3$ . Their preferences are as follows:

$$\begin{array}{ll}
s_1 : c_2 \succ c_1 \succ c_4 \succ c_5 & c_1 : \{s_1, s_2\} \succ \{s_1, s_3\} \succ s_1 \succ \{s_3, s_2\} \succ s_2 \succ s_3 \\
s_2 : c_1 \succ c_2 \succ c_3 & c_2 : s_3 \succ s_2 \succ s_1 \\
s_3 : c_1 \succ c_2 \succ c_3 & c_3 : s_3 \succ s_2 \\
s_4 : c_4 & c_4 : s_4 \succ s_1 \\
s_5 : c_5 & c_5 : s_5 \succ s_1.
\end{array}$$

In this case, if the schools have interim knowledge about capacity expansions, school  $c_1$  can manipulate the matching by reporting a false capacity of 0. Note that in this case, it is optimal to allocate the extra capacity to school  $c_1$ , obtaining the students-optimal stable matching  $\{(s_1, c_1), (s_2, c_3), (s_2, c_2), (s_4, c_4), (s_5, c_5)\}$  with an average rank of 9. Note that as a consequence of the manipulation school  $c_1$  improves her matching, which otherwise would be with students  $s_2, s_3$ . If there is no manipulation, every student would be matched to her most preferred school. Note that the preferences of the agents are responsive.  $\square$

*Proof of Proposition 2.* Recall function  $f : \mathbb{Z}_+^C \rightarrow \mathbb{R}_+$  defined for a given  $\mathbf{t} \in \mathbb{Z}_+^C$  as

$$f(\mathbf{t}) = \min \left\{ \sum_{(s,c) \in \mu} r_{s,c} : \mu \text{ is a stable matching in instance } \Gamma_{\mathbf{t}} \right\}.$$

First, we show that  $f$  is not submodular. Consider a set  $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$  of students and a set  $\mathcal{C} = \{c_1, c_2, c_3, c_4, c_5\}$  of schools. The preference lists are as follows

$$\begin{array}{ll}
s_1 : c_1, c_2, \dots & c_1 : s_1, s_2, \dots \\
s_2 : c_2, c_3, \dots & c_2 : s_2, s_3, \dots \\
s_3 : c_2, c_5, c_4, \dots & c_3 : s_2, s_1, \dots \\
s_4 : c_5, \dots & c_4 : s_3, s_4, \dots \\
& c_5 : s_4, s_3, \dots
\end{array}$$

School  $c_1$  has capacity 0 and the other schools have capacity 1. We choose the following two allocations:  $\mathbf{t} = (1, 0, 0, 0, 0)$  and  $\mathbf{t}' = (0, 1, 0, 0, 0)$ . Therefore, we obtain

$$f(\mathbf{t} \vee \mathbf{t}') + f(\mathbf{t} \wedge \mathbf{t}') = f(1, 1, 0, 0, 0) + f(0, 0, 0, 0, 0) = 4 > 3 = 2 + 1 = f(\mathbf{t}) + f(\mathbf{t}').$$

Second, we show that  $f$  is not supermodular. Consider a set  $\mathcal{S} = \{s_1, s_2, s_3\}$  of students and a set  $\mathcal{C} = \{c_1, c_2, c_3, c_4, c_5\}$  of schools. The preference lists are as follows

$$\begin{aligned} s_1 &: c_1, c_3, \dots & c_h &: s_1, s_2, \dots & \text{for all } h \in \{1, 2, 3\}. \\ s_2 &: c_2, c_4, \dots \\ s_3 &: c_3, c_4, c_5, \dots \\ s_4 &: c_5, \dots \end{aligned}$$

Schools  $c_1$  and  $c_2$  have capacity 0 and the other schools have capacity 1. We choose the following two allocations:  $\mathbf{t} = (1, 0, 0, 0, 0)$  and  $\mathbf{t}' = (0, 1, 0, 0, 0)$ . Therefore, we obtain

$$f(\mathbf{t} \vee \mathbf{t}') + f(\mathbf{t} \wedge \mathbf{t}') = f(1, 1, 0, 0, 0) + f(0, 0, 0, 0, 0) = 4 < 5 = 3 + 2 = f(\mathbf{t}) + f(\mathbf{t}').$$

□

The following example shows that the inverse inclusion of the statement in Theorem 5 does not necessarily hold.

EXAMPLE 2. Let us consider the set of students  $\mathcal{S} = \{s_1, s_2, s_3, s_4, s_5, s_6\}$  and the set of schools  $\mathcal{C} = \{c_1, c_2, c_3, c_4\}$ . The rankings of the students are as follows:

$$\begin{aligned} s_1 &: c_3, c_4, c_1, c_2 & c_1 &: s_1, s_3, s_2, s_5, s_6, s_4 \\ s_2 &: c_2, c_1, c_4, c_3 & c_2 &: s_4, s_1, s_6, s_5, s_2, s_3 \\ s_3 &: c_2, c_1, c_4, c_3 & c_3 &: s_2, s_1, s_5, s_6, s_3, s_4 \\ s_4 &: c_1, c_3, c_2, c_4 & c_4 &: s_6, s_3, s_2, s_4, s_5, s_1 \\ s_5 &: c_3, c_1, c_4, c_2 \\ s_6 &: c_1, c_3, c_4, c_2. \end{aligned}$$

Finally, schools' capacities are  $q_{c_1} = q_{c_2} = 1$  and  $q_{c_3} = q_{c_4} = 2$ . Given that we have to allocate optimally one extra position, the optimal solution for the relaxed aggregated linearization is  $x_{s_1c_3} = 1, x_{s_2c_2} = 1, x_{s_3c_1} = 0.16, x_{s_3c_2} = 0.66, x_{s_4c_1} = 0.5, x_{s_4c_2} = 0.16, x_{s_4c_3} = 0.16, x_{s_5c_3} = 0.83, x_{s_6c_1} = 0.5, x_{s_6c_4} = 0.33$  and  $\mathbf{t} = (0.16, 0.83, 0, 0)$  with the cost equal to 1.33.

On the other hand, the optimal solution for the relaxed non-aggregated linearization is  $x_{s_1c_3} = 0.83, x_{s_1c_4} = 0.08, x_{s_2c_2} = 1, x_{s_3c_1} = 0.11, x_{s_3c_2} = 0.66, x_{s_3c_4} = 0.02, x_{s_4c_1} = 0.66, x_{s_4c_3} = 0.16, x_{s_5c_3} = 1, x_{s_6c_1} = 0.52, x_{s_6c_4} = 0.30$  and  $\mathbf{t} = (0.30, 0.66, 0, 0.03)$  with cost equal to 1.03.

## Appendix B: Additional Background

### B.1. The Deferred Acceptance Algorithm

In this section, we recall the Deferred Acceptance algorithm introduced in Gale and Shapley (1962).

INPUT: An instance  $\Gamma = \langle \mathcal{S}, \mathcal{C}, \succ, \mathbf{q} \rangle$ .

OUTPUT: Student-oriented matching.

**Step 1:** Each student starts by applying to her most preferred school. Schools temporarily accept the most preferred applications and reject the less preferred applications which exceed their capacity.

**Step 2:** Each student  $s$  who has been rejected, proposes to her most preferred school to which she has not applied yet; if she has proposed to all schools, then she does not apply. If the capacity of the school is not met, then her application is temporarily accepted. Otherwise, if the school prefers her application to one of a student  $s'$  who was temporarily enrolled,  $s$  is temporarily accepted and  $s'$  is rejected. Vice-versa, if the school prefers all the students temporarily enrolled to  $s$ , then  $s$  is rejected.

**Step 3:** If all students are enrolled or have applied to all the schools they rank, return the current matching. Otherwise, go to Step 2.

## B.2. McCormick linearization

In this section we describe the McCormick convex envelope McCormick (1976) used to obtain a linear relaxation for bi-linear terms; if one of the terms is binary, the linearization provides an equivalent formulation. Consider a bi-linear term of the form  $x_i \cdot x_j$  with the following bounds for the variables  $x_i$  and  $x_j$ :  $l_i \leq x_i \leq u_i$  and  $l_j \leq x_j \leq u_j$ . Let us define  $y = x_i \cdot x_j$ ,  $m_i = (x_i - l_i)$ ,  $m_j = (x_j - l_j)$ ,  $n_i = (u_i - x_i)$  and  $n_j = (u_j - x_j)$ . Note that  $m_i \cdot m_j \geq 0$ , from which we derive the under-estimator  $y \geq x_i \cdot l_j + x_j \cdot l_i - l_i \cdot l_j$ . Similarly, it holds that  $n_i \cdot n_j \geq 0$ , from which we derive the under-estimator  $y \geq x_i \cdot u_j + x_j \cdot u_i - u_i \cdot u_j$ . Analogously, over-estimators of  $y$  can be defined. Make  $o_i = (u_i - x_i)$ ,  $o_j = (u_j - x_j)$ ,  $p_i = (x_i - l_i)$  and  $p_j = (x_j - l_j)$ . From  $o_i \cdot o_j \geq 0$  we obtain the over-estimator  $y \leq x_i \cdot l_j + x_j \cdot u_i - u_i \cdot l_j$ , and from  $p_i \cdot p_j \geq 0$  we obtain the over-estimator  $y \leq x_j \cdot l_i + x_i \cdot u_j - u_j \cdot l_i$ . The four inequalities provided by the over and under estimators of  $y$ , define the McCormick convex (relaxation) envelope of  $x_i \cdot x_j$ .

## Appendix C: Extensions

Our model can be easily extended to capture several relevant variants of the problem. In what follows we name some direct extensions:

- Adding budget: If there is a unit-cost  $p_c$  of increasing the capacity of school  $c$ , we can add an additional budget constraint of the form

$$\sum_{c \in \mathcal{C}} t_c \cdot p_c \leq B',$$

keeping all the other elements of the model unchanged. This extension could be used to allocate tuition waivers or other sort of scholarships that are school dependent.

- Different levels of granularity: Schools may not be free to expand their capacities by any value in  $\{1, \dots, B\}$ . This limitation can be easily incorporated into our model considering the unary expansion of the variables  $t_c$  for  $c \in \mathcal{C}$ . Specifically, let  $s_c^k = 1$  if the capacity of school  $c$  is expanded in  $k$  seats, and  $s_c^k = 0$  otherwise. Then, we know that  $\sum_{k=1}^B k \cdot s_c^k = t_c$ , and we must add the constraint that  $\sum_{k=1}^B s_c^k \leq 1$  for each  $c \in \mathcal{C}$ . Then, if the capacity of school  $c$  can only be expanded by values in a subset  $B' \subseteq B$ , we can enforce this by adding the constraints  $s_c^k = 0$  for all  $k \in B \setminus B'$ . This could also be captured using knapsack constraints.

- Adding secured enrollment: The Chilean system guarantees that students that are currently enrolled and apply to switch will be assigned to their current school if they are not assigned to a more preferred one.

This can be easily captured in our setting by introducing a parameter  $m_{s,c}$ , which is equal to 1 if student  $s$  is currently enrolled in school  $c$ , and 0 otherwise, and defining  $M = \{s \in \mathcal{S} : m_{s,c} = 1 \text{ for some } c \in \mathcal{C}\}$ . Then, we would only have to update a couple of constraints:

$$\begin{aligned} \sum_{s \in \mathcal{S}} x_{s,c} \cdot (1 - m_{s,c}) &\leq q_c + t_c, & \text{for all } c \in \mathcal{C}, \\ \sum_{c \in \mathcal{C}} x_{s,c} &= 1, & \text{for all } s \in M. \end{aligned} \tag{12}$$

The first constraint ensures that students currently enrolled do not count towards the capacity of the school they are currently enrolled. The second constraint ensures that all students that are currently enrolled are assigned to some schools (potentially, to the same school they are currently enrolled).

- **Room assignment:** Schools report the number of vacancies they have for each level. This decision depends on the classrooms they have and their capacity. However, schools decide (before the assignment) what level goes in each classroom, and this determines the number of reported vacancies for that level. This may introduce some inefficiencies, since some levels may be more demanded, and thus assigning a larger classroom may benefit both students in that school but also in others.

- **Quota assignment:** Many school choice systems have different quotas to serve under-represented students or special groups. For instance, in Chile there are quotas for low-income students (15% of total seats), for students with disabilities or special needs, and for students with high-academic performance. Moreover, some of these quotas may overlap, i.e., some students may be eligible for multiple quotas, and in most cases students count in only one of them. The number of seats available for each quota are pre-defined by each school, and schools have some freedom to define these quotas. Hence, our problem could be adapted to help schools define what is the best allocation of seats to quotas in order to improve students' welfare.

## Appendix D: Detailed Computational Results

In Table 3 we provide extensive computational results on the two heuristics, the IQP model and the aggregated linearization model. The columns with “gap” report the average percentage gap between the best upper bound found by the two exact methods and the solution found by the heuristic; low values mean better capacity to find a solution close to the optimum. When running the mathematical programs in Gurobi, we feed the best solution found by the heuristics as a warm start. The other columns contain the following information about each mathematical program: “rt-g” is the average percentage root gap between the root value and the best upper bound found by all methods; “nodes” is the average number of explored nodes; “#g” is the number of instances that reached the time limit of 1 hour; “%g” is the average last percentage gap found by Gurobi; “b-g” is the average gap between the upper bound found by the program and the best upper bound found by the two programs; “time” records the average time in seconds.

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S	C	B	LPH	Greedy	Integer Quadratic Program (IQP)						Aggregated linearization (Agg-Lin)					
			gap	gap	rt-g	nodes	#g	%g	b-g	time	rt-g	nodes	#g	%g	b-g	time
1000	5	1	1.4	0.0	44.7	110.6	0	0.0	0.0	21.75	54.0	315.5	0	0.0	0.0	26.23
		2	2.9	0.3	51.5	373.4	0	0.0	0.0	14.23	50.7	107.3	0	0.0	0.0	22.19
		5	6.0	3.6	44.1	126.5	0	0.0	0.0	10.57	43.9	89.1	0	0.0	0.0	21.05
		10	11.8	11.0	41.6	146.8	0	0.0	0.0	11.02	41.6	172.6	0	0.0	0.0	22.93
		20	18.0	28.9	28.3	21.6	0	0.0	0.0	7.95	28.3	102.2	0	0.0	0.0	17.91
		30	25.2	59.1	18.0	15.1	0	0.0	0.0	5.65	18.0	158.7	0	0.0	0.0	12.93
1000	8	1	1.6	0.0	66.2	25068.6	4	2.1	0.0	1488.81	64.7	1929.9	0	0.0	0.0	181.38
		2	3.2	0.8	61.1	819.8	0	0.0	0.0	60.67	60.6	1324.9	0	0.0	0.0	146.83
		5	6.0	3.1	58.6	653.4	0	0.0	0.0	56.41	58.4	1185.1	0	0.0	0.0	111.91
		10	10.6	8.3	55.7	620.9	0	0.0	0.0	60.73	55.6	1204.2	0	0.0	0.0	108.18
		20	18.9	18.3	44.6	347.0	0	0.0	0.0	64.27	44.6	702.1	0	0.0	0.0	63.78
		30	31.7	51.0	27.7	77.6	0	0.0	0.0	22.52	27.7	432.7	0	0.0	0.0	33.15
1000	10	1	2.1	0.0	66.7	27688.8	13	9.5	0.0	2427.38	65.6	8646.9	6	5.2	0.0	1324.37
		2	2.8	0.9	65.8	1015.3	0	0.0	0.0	207.02	64.9	1825.4	0	0.0	0.0	331.98
		5	5.8	3.7	62.7	956.3	0	0.0	0.0	170.84	62.4	1635.7	0	0.0	0.0	227.79
		10	10.0	8.2	58.3	819.7	0	0.0	0.0	308.82	58.1	1395.7	0	0.0	0.0	187.56
		20	20.4	19.9	44.9	464.7	0	0.0	0.0	148.93	44.9	772.6	0	0.0	0.0	112.19
		30	34.7	51.4	32.1	42.2	1	0.1	0.0	44.59	32.1	436.3	0	0.0	0.0	64.01
1000	15	1	2.2	0.0	71.8	11773.6	28	29.5	0.0	3519.41	70.9	6765.4	27	30.5	0.0	3426.71
		2	4.2	0.8	70.2	1699.7	0	0.0	0.0	1427.29	69.2	2352.5	1	0.5	0.0	964.12
		5	7.8	3.6	67.4	2070.9	1	0.1	0.0	1499.12	66.7	3007.2	0	0.0	0.0	822.86
		10	12.7	8.9	65.6	1802.3	1	0.6	0.0	1442.69	65.3	3305.5	0	0.0	0.0	710.30
		20	23.3	20.1	51.4	864.2	0	0.0	0.0	652.50	51.4	1480.8	0	0.0	0.0	349.08
		30	31.4	32.7	44.3	562.8	0	0.0	0.0	450.67	44.3	1134.5	0	0.0	0.0	248.97
2000	5	1	0.6	0.0	59.4	9444.3	0	0.0	0.0	538.29	57.5	1473.8	0	0.0	0.0	108.17
		2	2.1	0.3	54.4	1691.5	0	0.0	0.0	81.46	53.3	356.6	1	0.0	0.0	117.00
		5	4.3	2.7	56.2	1762.2	0	0.0	0.0	59.17	55.9	716.3	0	0.0	0.0	139.85
		10	8.0	5.6	49.2	1370.4	0	0.0	0.0	58.33	49.0	611.6	0	0.0	0.0	137.41
		20	10.0	16.4	41.6	889.6	0	0.0	0.0	53.26	41.6	434.3	0	0.0	0.0	115.28
		30	11.4	33.4	25.3	187.7	0	0.0	0.0	36.10	25.3	230.7	0	0.0	0.0	92.03
2000	8	1	1.3	0.0	67.4	11057.6	17	17.4	0.0	2793.14	66.7	3856.3	7	5.3	0.0	1665.35
		2	2.5	0.6	67.0	17375.8	2	0.5	0.0	1386.71	66.7	2094.3	0	0.0	0.0	861.73
		5	5.0	2.8	62.5	12946.5	4	1.8	0.0	1211.51	62.4	2911.9	2	0.4	0.0	1059.36
		10	8.5	4.9	60.7	13625.2	1	0.3	0.0	944.79	60.7	3414.9	2	0.6	0.0	1099.34
		20	18.0	15.9	48.9	9011.9	2	0.8	0.0	817.34	48.9	2365.6	0	0.0	0.0	666.67
		30	17.5	23.3	44.3	6149.3	1	0.4	0.0	391.54	44.3	2442.0	1	0.4	0.0	410.77
2000	10	1	1.0	0.0	71.0	1355.9	29	38.8	0.0	3528.57	70.7	3569.4	22	25.7	0.0	3140.23
		2	2.4	0.6	68.9	24435.4	21	16.0	0.0	3017.29	68.7	3393.4	4	2.4	0.0	1853.09
		5	4.2	2.3	67.4	18647.5	19	16.3	0.0	3066.72	67.3	3610.9	10	7.1	0.0	2270.44
		10	7.1	5.8	60.7	17816.0	9	8.0	0.1	2248.52	60.8	4481.9	6	3.8	0.0	1904.39
		20	13.9	10.9	56.9	9829.0	13	10.6	0.0	2146.54	56.9	3549.1	6	3.3	0.1	1640.89
		30	19.1	21.8	46.2	5591.9	7	6.0	0.2	1394.91	46.2	3359.9	1	0.5	0.0	830.71
2000	15	1	1.3	0.0	73.3	5793.4	30	54.4	0.0	3600.18	73.2	1424.3	30	48.5	0.0	3601.82
		2	2.5	0.6	72.3	2563.4	30	47.9	0.2	3600.14	72.2	2793.9	26	22.3	0.0	3514.34
		5	5.1	2.8	69.1	1680.6	30	29.9	0.1	3600.63	69.0	3708.1	22	18.4	0.1	3347.38
		10	8.4	6.2	63.9	649.9	26	22.1	0.4	3349.68	64.0	4909.3	13	9.9	0.3	2833.55
		20	14.7	14.2	56.1	535.1	21	15.0	1.8	2888.56	56.1	4524.6	8	4.3	0.0	2140.14
		30	21.1	25.0	48.6	506.1	14	7.0	1.4	2359.37	48.6	5229.9	7	4.9	0.8	1815.62

Table 3 Average results for each triplet  $|S|$ ,  $|C|$  and  $B$ .

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